PT = CHARACTERISTIC FUNCTIONS

Useful tools in probability theory include integral transforms. They all take the form $\mathbb{E}[\square^X]$; X = Random Variable

integral ... something raised to the power X.

Ex: • $\square = s$; $|s| \leq 1$: GENERATING FUNCTIONS

$g_X(s) = \mathbb{E}[s^X]$ is commonly used for discrete random variables

• $\square = e^t$: MOMENT GENERATING FUNCTIONS

As the exponential function is unbounded, $m(t) = \mathbb{E}[e^{tX}]$ may be infinite. For example, if $X \sim Exp(\lambda)$, then $m_X(t)$ is infinite for $t \geq \lambda$.

• $\square = e^{-t}$: LAPLACE TRANSFORM

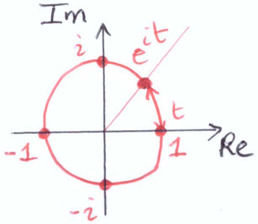
Used mostly for non-negative RVs since $\ell_X(t) = \mathbb{E}[e^{-tX}] < \infty$, always.

We focus on Characteristic Functions (ChF), for which

• $\square = e^{it} = \cos t + i \sin t$ (Euler's formula)

$\varphi_X(t) = \mathbb{E}[e^{itX}]$
$= \mathbb{E}(\cos tX) + i\, \mathbb{E}(\sin tX)$

Complex-valued



$\varphi_X(t)$ always exists and is finite,
$|\varphi_X(t)| = |\mathbb{E}\, e^{itX}| \leq \mathbb{E}\, |e^{itX}| = 1$

Moreover, $\varphi_X(0) = \mathbb{E}\, e^{it0} = 1$.

Thus, • $\varphi_X(t) = \sum_k e^{itk}\, \mathbb{P}(X=k)$ for $X \in \mathbb{Z}$

• $\varphi_X(t) = \underbrace{\int e^{itx} f(x)\,dx}$ for X = AC.

This expression is (up to some constant) the Fourier transform of the density $f(x)$.

Theorem: The ChF specifies the distribution of X: If $\varphi_X = \varphi_Y$, then X and Y have the same distribution function, and vice-versa.

↳ Properties of characteristic functions.

$\boxed{[P1]\ \varphi_X(-t) = \overline{\varphi_X(t)}}$ ← complex conjugate: $\overline{x+iy} = x - iy$

This follows from

$\varphi_X(-t) = \mathbb{E}[e^{-itX}] = \mathbb{E}[\cos(-tX)] + i\,\mathbb{E}[\sin(-tX)]$
$= \mathbb{E}[\cos(tX)] - i\,\mathbb{E}[\sin(tX)]$
$= \overline{\varphi_X(t)}$

Corollary: The distribution of X is symmetric: $X \overset{d}{=} -X$
$\Updownarrow$
$\varphi_X(t)$ is real-valued

proof $\boxed{\Downarrow}$ Suppose $X \overset{d}{=} -X$.
Then $\varphi_X(t) = \varphi_{-X}(t) = \mathbb{E}\, e^{it(-X)} = \varphi_X(-t) = \overline{\varphi_X(t)}$

$\Rightarrow \varphi_X(t) \in \mathbb{R}$.

$\boxed{\Uparrow}$ If $\varphi_X(t) \in \mathbb{R}$, then
$\varphi_X(t) = \overline{\varphi_X(t)} = \varphi_X(-t) = \mathbb{E}\, e^{i(-t)X} = \varphi_{-X}(t)$ ■

[P2] $\varphi_{aX+b}(t) = e^{ibt}\varphi_X(at)$ $\quad \forall a,b \in \mathbb{R}$

Indeed, $\varphi_{aX+b}(t) = \mathbb{E}[\exp(it(aX+b))]$

$$= \mathbb{E}[e^{itb} e^{i(ta)X}]$$

$$= e^{itb}\varphi_X(at) \quad \blacksquare$$

[P3] If X and Y are independent, $\varphi_{X+Y}(t) = \varphi_X(t)\varphi_Y(t)$

[P4] Any ChF is uniformly continuous.

Recall that a function f is uniformly continuous if

$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x,y \ |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

does not depend on x and y.

Recall that:

Continuously differentiable $\subset$ Lipschitz continuous $\subset$ Uniformly continuous $\subset$ Continuous

Limited in how fast it can fluctuates: $|f(x) - f(y)| \leq c|x-y|$

proof: let $\varepsilon > 0$, and put $I_n := [-n, n]$.
Then, $\forall t, h \in \mathbb{R}$,

$$|\varphi_X(t+h) - \varphi_X(t)| = \left|\mathbb{E}[e^{i(t+h)X}] - \mathbb{E}[e^{itX}]\right|$$

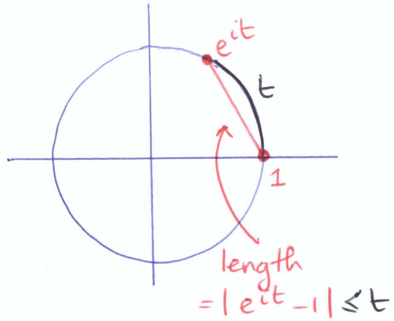
$$= \left|\mathbb{E}[e^{i(t+h)X} - e^{itX}]\right|$$

$$= \left|\mathbb{E}[e^{itX}(e^{ihX} - 1)]\right|$$

$$\leq \mathbb{E}\underbrace{|e^{itX}|}_{=1}|e^{ihX} - 1|$$

$$= \mathbb{E}|e^{ihX} - 1|$$

$$|\varphi_X(t+h) - \varphi_X(t)| = \mathbb{E}\left(|e^{ihX} - 1|; X \in I_n\right) + \underbrace{\mathbb{E}\left(|e^{ihX} - 1|; X \notin I_n\right)}_{\leq 2\mathbb{P}(X \notin I_n)}$$



$|e^{ihX} - 1| \leq |hX|$
$\leq |h|n$ on I_n.

Thus

$$|\varphi_X(t+h) - \varphi_X(t)| \leq |h|n + 2\mathbb{P}(X \notin I_n).$$

Choose n large enough so that this $\leq \varepsilon/2$.

Then the RHS is $< \varepsilon$ for $|h| < \frac{\varepsilon}{2n}$, independently of the value of t.

[P5]
- If $\mathbb{E}|X| < \infty$, then $\varphi_X(t)$ is continuously differentiable and $\varphi'_X(0) = i\mathbb{E}X$.
- If $\mathbb{E}|X|^k < \infty$, then $\varphi_X(t)$ is k times continuously differentiable, and $\varphi_X^{(k)}(0) = i^k \mathbb{E}X^k$

proof: Let's compute

$$\lim_{h \to 0} \frac{\varphi_X(h) - \varphi_X(0)}{h} = \lim_{h \to 0} \int \left\{\frac{e^{ihu} - 1}{h}\right\} dF_X(u)$$

limit does not necessarily exist

To exchange lim and $\int$, we make use of the Dominated Convergence Theorem (DCT).

Since $\left| \frac{e^{ihu}-1}{h} \right| \leq \frac{|hu|}{|h|} = |u|$, and by assumption $\int |u| \, dF_X(u) = \mathbb{E}|X| < \infty$, we get

$$\lim_{h\to 0} \frac{\varphi_X(h) - \varphi_X(0)}{h} = \int \lim_{h \to 0} \left\{ \frac{e^{ihu}-1}{h} \right\} dF_X(u)$$

$$= \int \lim_{h\to 0} \left\{ \frac{\cos hu + i \sin hu - 1}{h} \right\} dF_X(u)$$

$$= i \int u \, dF_X(u) = i \mathbb{E} X .$$

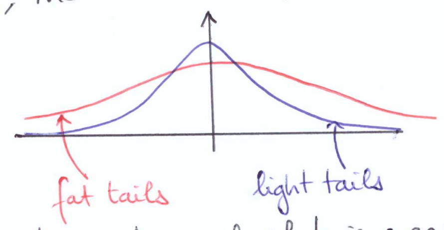
so the limit exists and $\varphi_X'(0) = i\mathbb{E}X$.

The case $X \in \mathcal{L}^k$ is treated similarly. ■

↳ In other words, if $X \in \mathcal{L}^2$, then $\varphi_X(t)$ is twice continuously differentiable, and

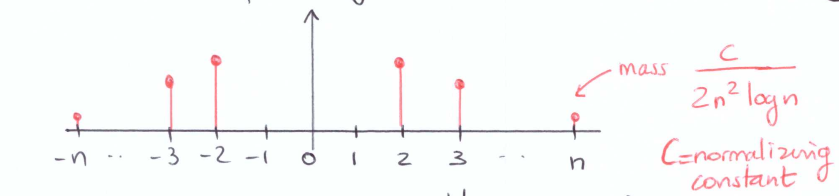
$$\varphi_X(t) = 1 + it\mathbb{E}X - \frac{1}{2} t^2 \mathbb{E}X^2 + o(t^2) .$$

⇒ "The lighter the tails of F_X [≡ the more finite moments X has], the smoother $\varphi_X(t)$"



The converse is almost true: true only if k is even: If $\varphi_X(t)$ is twice differentiable at zero, then $\mathbb{E}X^2 < \infty$. It can fail if k is odd. We better see this through an example: we present a distribution for which $\varphi_X'(0)$ exists, but $\mathbb{E}|X| = \infty$.

→ Consider the following distribution:



So that $\varphi_X(t) = \sum_k e^{itk} \mathbb{P}(X=k)$

$$= \sum_{k\geq 2} e^{itk} \frac{c}{2k^2 \log k} + \sum_{k \geq 2} e^{-itk} \frac{c}{2k^2 \log k}$$

$$= \sum_{k\geq 2} \frac{C \cos(kt)}{k^2 \log k} .$$

• $\mathbb{E}|X| = \sum_{k\geq 2} \frac{2|k| C}{2k \log k} = C \sum_{k\geq 2} \frac{1}{k \log k} = \infty$

Integral test: the sum converges iff the integral $\int_2^\infty \frac{dx}{x \log x} < \infty$.

Since $\int_2^t \frac{dx}{x \log x} = \left[\log(\log x) \right]_2^t \to \infty$ as $t \to \infty$, the infinite series diverges as well.

• However, $\varphi_X(t) = \sum_{k \geq 2} \underbrace{\frac{C\cos(kt)}{k^2 \log k}}_{|\cdots| \leq \frac{c}{k^2 \log k}}$

and $c \sum_{k\geq 2} \frac{1}{k^2 \log k} < \infty$

⇒ Weierstrass-M-test for infinite series implies that the series is uniformly convergent. Moreover, the series of termwise derivatives is also uniformly convergent (see for instance Theorem 1.3 p. 182

in Zygmund (2002) – Trigonometric series. Cambridge University Press, 3rd Edition). Since a uniform series of differentiable functions can be differentiated term by term, provided that the derivative series converges uniformly (see for example Theorem 10 p. 210 in Pugh (2002) – Real Mathematical analysis. Springer), we get that $\varphi'_X(t) = -C \sum_{k\geq 2} \frac{\sin(kt)}{k\log k}$. In particular, we see that $\varphi'_X(0) < \infty$.

#TakeAway

Summarizing, the lighter the tails of F_X, the smoother φ_X; and [almost] the other way around.

1 Say if the given functions are the ChF of some distribution. Explain / Justify your answers

(i) $\frac{1}{3}e^{-|t|} + \frac{2}{3}e^{-t^2}$

(ii) $\cos^3 t$

(iii) $\sin t$

(iv) $\frac{\sin t^2}{t^2}$

(v) $\frac{1}{2} + \frac{1}{2}\cos t$

Examples: (i) Binomial distribution.

Suppose first that $X \sim B(p)$. Then

$$\varphi_X(t) = \mathbb{E}\left[e^{itX}\right] = p\, e^{it1} + q\, e^{it0} = pe^{it} + q$$

$q := 1-p$

Now, since $Y := \underbrace{X_1 + \cdots + X_n}_{\text{i.i.d} \sim B(p)} \sim Bi(n,p)$

$$\underset{\text{indpce}}{\varphi_Y(t) =} \left[\varphi_{X_1}(t)\right]^n = \left(pe^{it} + q\right)^n = \text{complex function.}$$

Moreover,
- $\varphi_Y(0) = 1$
- $\varphi'_Y(0) = inp$
- $\varphi''_Y(0) = -npq - n^2p^2$ …/…

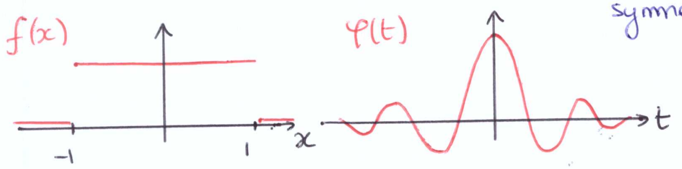
(ii) Poisson distribution. $X \sim \mathcal{P}(\lambda)$

$$\varphi_X(t) = e^{-\lambda}\sum_{k\geq 0} e^{itk}\frac{\lambda^k}{k!} = e^{-\lambda}\sum_{k\geq 0}\frac{\left[e^{it}\lambda\right]^k}{k!} = \exp\left[\lambda\left(e^{it}-1\right)\right].$$

(iii) Uniform distribution $X \sim \mathcal{U}(-1,1)$.

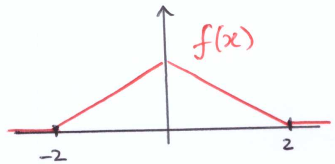
Then
$$\varphi_X(t) = \int e^{itx} f(x)\,dx = \int_{-1}^{1}\frac{1}{2}e^{itx}\,dx = \frac{1}{2}\left[\frac{1}{it}e^{itx}\right]_{-1}^{1} = \frac{e^{it}-e^{-it}}{2it} = \frac{\sin t}{t} \quad (\varphi_X(0) = 1)$$

real and symmetric.



2 (i) Show that the function
$$f(x) = \frac{1}{2}\left(1 - \frac{|x|}{2}\right)\mathbb{1}(|x|<2)$$
corresponds to the density of $U_1 + U_2$, where U_1 and U_2 are independly $\mathcal{U}(-1,1)$ distributed.

(ii) Compute the ChF of $U_1 + U_2$ ($\equiv$ Fourier transform of f).



(iv) Normal distribution.

let $Z \sim \mathcal{N}(0,1)$. Then

$$\varphi_Z(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{itx} \, e^{-x^2/2} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (\cos tx) \, e^{-x^2/2} dx + \underbrace{\frac{i}{\sqrt{2\pi}} \int (\sin tx) \, e^{-x^2/2} dx}_{=0}$$

Differentiating under the integral sign,

$$\varphi_Z'(t) = -\frac{1}{\sqrt{2\pi}} \int x \, e^{-x^2/2} \sin(tx) \, dx$$

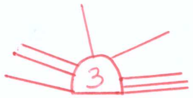
[which is justified since $|x e^{-x^2/2} \sin tx|$ is integrable]

Integration by parts.

$$= -\frac{1}{\sqrt{2\pi}} \int t \cos(tx) e^{-x^2/2} \, dx$$

$$= -t \, \varphi_Z(t)$$

$\Rightarrow \varphi_Z$ is solution to the differential equation $\varphi_Z'(t) = -t \, \varphi_Z(t)$, with initial condition $\varphi_Z(0) = 1$. $\Rightarrow \varphi_Z(t) = e^{-t^2/2}$.



Alternatively, show that $\varphi_Z(t) = \frac{e^{-t^2/2}}{\sqrt{2\pi}} \int e^{-\frac{1}{2}(x-it)^2} dx$, and compute the integral using contour integration (Use Cauchy Theorem).

Note that $\varphi_Z'(0) = 0$; $\varphi_Z''(0) = -\mathbb{E} Z^2 = -1$.

For $X = \mu + \sigma Z \sim \mathcal{N}(\mu, \sigma^2)$, we get

$$\boxed{\varphi_X(t) = \exp\left\{ it\mu - \frac{\sigma^2 t^2}{2} \right\}}$$

Remarks: (i) ChF of a multivariate normal distribution.

For $\underline{X} \sim \mathcal{N}(\underline{\mu}, \Sigma)$, $\underline{\mu} \in \mathbb{R}^d$, $\Sigma \in \mathbb{R}^{d \times d}$.

$$f(\underline{x}) = \frac{1}{(2\pi)^{d/2} \sqrt{|\Sigma|}} \exp\left\{ -\frac{1}{2} (\underline{x} - \underline{\mu})^t \Sigma^{-1} (\underline{x} - \underline{\mu}) \right\}$$

The ChF of a random vector $\underline{X} \in \mathbb{R}^d$ is defined as $\varphi_{\underline{X}}(\underline{t}) = \mathbb{E}\left[\exp(i \, \underline{t}^t \underline{X}) \right]$,

where $\underline{t}^t \underline{X} = \langle \underline{t}, \underline{X} \rangle = \sum_{i=1}^d t_i X_i$,

$\underline{t} = (t_1, \dots, t_d)^t \in \mathbb{R}^d$

$\underline{X} = (X_1, \dots, X_d) \in \mathbb{R}^d$.

Then $\boxed{\varphi_{\underline{X}}(\underline{t}) = \exp\left\{ i \, \underline{t}^t \underline{\mu} - \frac{1}{2} \underline{t}^t \Sigma \, \underline{t} \right\}}$.

Indeed, $\underline{X} \in \mathbb{R}^d$ has a multivariate normal distribution if and only if any linear combination $\sum_{i=1}^d \lambda_i X_i$, $\lambda_i \in \mathbb{R}$, is a (univariate) normal RV.

$\Rightarrow \mathbb{E} \langle \underline{\lambda}, \underline{X} \rangle = \mathbb{E} \, \underline{\lambda}^t \underline{X} = \underline{\lambda}^t \underline{\mu}$, $\underline{\lambda} = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_d \end{pmatrix}$

$\mathrm{Var} \langle \underline{\lambda}, \underline{X} \rangle = \underline{\lambda}^t \Sigma \, \underline{\lambda}$

$\Rightarrow$ The random variable $\langle \underline{\lambda}, \underline{X} \rangle \sim \mathcal{N}(\underline{\lambda}^t \underline{\mu}, \underline{\lambda}^t \Sigma \underline{\lambda})$, and its ChF is given by

$$\varphi_{\langle \underline{\lambda}, \underline{X} \rangle}(\underbrace{t}_{\in \mathbb{R}}) = \mathbb{E}\left\{ e^{it \langle \underline{\lambda}, \underline{X} \rangle} \right\}$$

$$= \exp\left\{ it(\underline{\lambda}^t \underline{\mu}) - \frac{1}{2} t^2 (\underline{\lambda}^t \Sigma \underline{\lambda}) \right\}.$$

$\Rightarrow$ We deduce the ChF of $\underline{X}$ by replacing t by 1 and $\underline{\lambda}$ by $\underline{t}$:

$$\varphi_{\underline{X}}(\underline{t}) = \mathbb{E}\left\{ e^{i \langle \underline{t}, \underline{X} \rangle} \right\} = \exp\left\{ i \, \underline{t}^t \underline{\mu} - \frac{1}{2} \underline{t}^t \Sigma \, \underline{t} \right\} \quad \blacksquare$$

(ii) Notice that for the $\mathcal{U}(-1,1)$ and $\mathcal{N}(\mu,\sigma^2)$ distributions, $|\varphi(t)| \to 0$ as $t \to \pm\infty$.

$$\left(\begin{array}{l} \varphi(t) = \frac{\sin t}{t} \text{ for } \mathcal{U}(-1,1) \\ \varphi(t) = \exp\left\{it\mu - \frac{t^2\sigma^2}{2}\right\} \text{ for } \mathcal{N}(\mu,\sigma^2) \end{array} \right)$$

It turns out that necessarily for AC random variables, holds $\lim_{t\to\pm\infty} |\varphi(t)| = 0$ [a consequence of the Riemann-Lebesgue Theorem for Fourier transforms]

Ex: Suppose that the density f is continuously differentiable on some interval $[a,b] \subset \mathbb{R}$. Then

$$\int_a^b e^{itx} f(x)\,dx = \frac{1}{it} f(b) e^{itb} - \frac{1}{it} f(a) e^{ita} - \frac{1}{it}\int_a^b e^{itx} f'(x)\,dx.$$

integrating by parts

and indeed the RHS $\to 0$ as $t \to \pm\infty$.

We mentioned previously that the ChF uniquely specifies the distribution. In fact, there is more: under integrability conditions, it is possible to explicitely recover the expression of the distribution from the ChF:

INVERSION FORMULA

> If $\int |\varphi_X(t)|\,dt < \infty$, then X has a continuous density given by
> $$f_X(x) = \frac{1}{2\pi}\int e^{-itx} \varphi_X(t)\,dt$$

However, we can rarely invert analytically a ChF ...

Analogy with Fourier Series

- For an AC random variable X with (integrable) density f (since $\int_{\mathbb{R}} f(x)\,dx = 1$), the Fourier transform is well-defined and given by $\varphi(t) = \int_{\mathbb{R}} e^{itx} f(x)\,dx$.

- Suppose now that $f \in L^2(0,T) = \left\{ f \mid \int_0^T |f(x)|^2 dx < \infty \right\}$.

Then the functions
$$\begin{cases} f_0(x) = \frac{1}{\sqrt{T}} \\ f_k(x) = \sqrt{\frac{2}{T}} \cos\left(\frac{2\pi k x}{T}\right), & i \geq 1 \\ g_k(x) = \sqrt{\frac{2}{T}} \sin\left(\frac{2\pi k x}{T}\right), & i \geq 1 \end{cases}$$

form an orthonormal base of $L^2(0,T)$ [endowed with the inner product $\langle u, v \rangle = \int_0^T u(x) v(x)\,dx$]

$\Rightarrow$ Any function $f \in L^2(0,T)$ is the limit in $L^2(0,T)$ of
$$\hat{f}_n = \langle f, f_0 \rangle f_0 + \langle f, f_1 \rangle f_1 + \dots + \langle f, f_n \rangle f_n + \langle f, g_1 \rangle g_1 + \dots + \langle f, g_n \rangle g_n,$$
in the sense that $\|f - \hat{f}_n\| \to 0$ as $n \to \infty$, $\|.\|$ being the norm induced by the inner product $\langle .,. \rangle$.

Re-writing this expression slightly differently,

- $\langle f, f_0 \rangle f_0 = \frac{1}{T}\int_0^T f(x)\,dx =: a_0$
- $\langle f, f_k \rangle f_k = \underbrace{\frac{2}{T}\left\{\int_0^T f(x)\cos\left(\frac{2\pi k x}{T}\right)dx\right\}}_{=: a_k} \cos\left(\frac{2\pi k x}{T}\right)$
- $\langle f, g_k \rangle g_k = \underbrace{\frac{2}{T}\left\{\int_0^T f(x)\sin\left(\frac{2\pi k x}{T}\right)dx\right\}}_{=: b_k} \sin\left(\frac{2\pi k x}{T}\right)$

Putting $\begin{cases} h_0(x) = a_0 \\ h_k(x) = a_k \cos\left(\frac{2\pi k x}{T}\right) + b_k \sin\left(\frac{2\pi k x}{T}\right) \end{cases}$,

we obtain $\hat{f}_n(x) = \sum_{k=0}^{n} h_k(x) \xrightarrow{L^2(0,T)} f$, as $n \to \infty$

superposition of pure oscillations vibrating at different frequencies [the higher the index i, the higher the frequency]

This expansion can be re-expressed in terms of complex, by replacing the $\cos\alpha$ terms by $\frac{1}{2}(e^{i\alpha} + e^{-i\alpha})$ and the $\sin\alpha$ terms by $\frac{1}{2i}(e^{i\alpha} - e^{-i\alpha})$:

$$h_k(x) = c_k \exp\left(\frac{2i\pi k x}{T}\right) + c_{-k} \exp\left(-\frac{2i\pi k x}{T}\right),$$

where $\begin{cases} c_k := \frac{a_k - i b_k}{2}, & k > 0 \\ c_{-k} := \frac{a_k + i b_k}{2} = \bar{c}_k, & k > 0 \\ c_0 := a_0 \end{cases}$

we see that $\hat{f}_n(x) = \sum_{k=-n}^{n} \bar{c}_k \exp\left(\frac{2i\pi k x}{T}\right)$

⇒ FOURIER SERIES of f is $\sum_{k=-\infty}^{+\infty} d_k \exp\left(\frac{2i\pi k x}{T}\right)$,

where $d_k = \frac{1}{T}\int_0^T f(x) \exp\left(-\frac{2i\pi k x}{T}\right) dx$, $k \in \mathbb{Z}$

To obtain the expression of d_k, check that the expression is valid for c_k, $k>0$, then for c_{-k}, $k>0$ as well, noticing that $c_{-k} = \bar{c}_k$

Now, compare the expression of the Fourier series of $f \in L^2(0,T)$, and the ChF of an AC random variable X:

FOURIER SERIES	FOURIER TRANSFORM / ChF
$f \in L^2(0,T)$	$f \in L^1(\mathbb{R})$
$\underbrace{f(x) = \sum_{k\in\mathbb{Z}}}_{\text{superposition of pure frequencies}} \underbrace{d_k}_{\text{strength of the oscillation at frequency } \frac{k}{T} \text{ (period } \frac{T}{k})} \underbrace{e^{\frac{2i\pi k x}{T}}}_{\text{oscillation}}$	$f(x) = \frac{1}{2\pi}\int \underbrace{\varphi(t)}_{\text{contribution of oscillations at "frequency" } t} \underbrace{e^{-itx}}_{\text{oscillation}} dt$ (inversion formula)
$d_k = \frac{1}{T}\int_0^T f(x) e^{-\frac{2i\pi k x}{T}} dx$	$\varphi(t) = \int f(x) e^{itx} dx$

⇒ Same story! Simply "replace" sums by integrals.

Examples: (i) We cannot apply the inversion formula to $\varphi_X(t) = \frac{\sin t}{t}$ since $\int \left|\frac{\sin t}{t}\right| dt = \infty$; although X has a density. However, this is not surprising as the density is discontinuous.

(ii) For $\varphi_X(t) = e^{-t^2/2}$, we get

$$f(x) = \frac{1}{2\pi}\int e^{-itx} e^{-t^2/2} dt$$

$$= \frac{1}{\sqrt{2\pi}} \underbrace{\frac{1}{\sqrt{2\pi}} \int e^{it(-x)} e^{-t^2/2} dt}$$

Compare with the expression page 9: the role of t and x are inverted and this term is equal to $\varphi_Z(-x) = e^{-\frac{x^2}{2}}$.

(iii) ChF of $X \sim \text{Exp}(1)$

$$\varphi_X(t) = \int_0^\infty e^{itx} e^{-x}\, dx = \int_0^\infty e^{-(1-it)x}\, dx$$

$$= -\frac{1}{1-it}\left[e^{-(1-it)x}\right]_0^\infty$$

$$= \frac{1}{1-it}.$$

Not allowed to apply the inversion formula here, as $\int |\varphi_X(t)|dt = \infty$. Indeed, X has a discontinuous density (at 0).

(iv) Consider the double exponential function

$$f_X(x) = \frac{1}{2}e^{-|x|}, \quad x \in \mathbb{R}.$$

(= a mixture of Exp(1) and its reflection).

$$\varphi_X(t) = \int e^{itx}\left\{\frac{1}{2}f_Y(x) + \frac{1}{2}f_{-Y}(x)\right\}dx$$

$f_Y(x) = e^{-x}\mathbb{1}(x>0)$ $\quad Y \sim \text{Exp}(1)$

& $f_{-Y}(x) = e^{x}\mathbb{1}(x<0)$

$$= \frac{1}{2}\left(\varphi_Y(t) + \underbrace{\varphi_{-Y}(t)}_{=\varphi_Y(-t)}\right)$$

$$= \frac{1}{2}\left(\frac{1}{1-it} + \frac{1}{1+it}\right)$$

$$\varphi_X(t) = \frac{1}{1+t^2} = \text{a real \& symmetric function.}$$

$$= \left(\frac{1}{1-it}\right)\left(\frac{1}{1+it}\right)$$

$$= \varphi_{Y_1}(t)\,\varphi_{-Y_2}(t), \quad Y_1, Y_2 \text{ indpt} \sim \text{Exp}(1)$$

easier than using the convolution formula

so that we see that $X \overset{d}{=} Y_1 - Y_2$.

Since $\int \varphi_X(t)\,dt < \infty$, we can use the inversion formula:

$$f_X(x) = \frac{1}{2}e^{-|x|} = \frac{1}{2\pi}\int \frac{e^{-itx}}{1+t^2}\,dt$$

↓ Replace x by t

$$\cancel{\frac{1}{2}}e^{-|t|} = \frac{1}{\cancel{2}\pi}\int \frac{e^{-itx}}{1+x^2}\,dx$$

↓ Replace t by $-t$

$$e^{-|t|} = \int \frac{e^{itx}}{\pi(1+x^2)}\,dx$$

$$= \int g(x)\, e^{itx}\,dx,$$

where $g(x) = \dfrac{1}{\pi(1+x^2)}$ = density of a Cauchy RV.

$\Rightarrow$ the ChF of a Cauchy distribution is $e^{-|t|}$.

Remark: Just by looking at $e^{-|t|}$, we can deduce that the associated RV is not integrable (since the ChF is not differentiable at 0), and has a symmetric density (since the ChF is real).

Remark: We established on page 7 the correspondance:

$$\boxed{\text{smoother } \varphi_X(t) \iff \text{lighter the tails of } f_X(x)}$$

Using a similar argument as on page 4/5, one can show that if $\int |t^k \varphi_X(t)|\,dt < \infty$, then X has a k-times continuously differentiable density, so that

$$\boxed{\text{lighter the tails of } \varphi_X(t) \iff \text{smoother } f_X(x)}$$

4. Check that the claim is true for $k=1$:
If $\int |t\,\varphi_X(t)|\,dt < \infty$, then X has a continuously differentiable density.

Consequences:

If X_1 and X_2 are independent RVs, then the ChF of $X_1 + X_2$ is

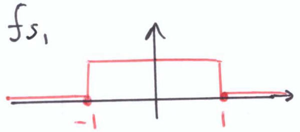
$$\varphi_{X_1+X_2}(t) = \underbrace{\varphi_{X_1}(t)\,\varphi_{X_2}(t)}_{\text{decay towards } 0 \text{ as } t \to \pm\infty}$$

the product decays to 0 faster than any of the $\varphi_{X_i}(t)$

$\Rightarrow$ the tails of $\varphi_{X_1+X_2}(t)$ are lighter than the tails of $\varphi_{X_1}(t)$ or $\varphi_{X_2}(t)$ $\Rightarrow$ the distribution of $X_1 + X_2$ is smoother than the distribution of X_1 or X_2.

Ex: $S_n = X_1 + \cdots + X_n$, where $X_i \sim \mathcal{U}(-1, 1)$

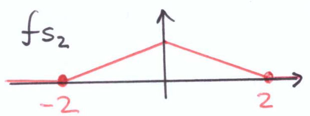
f_{S_1} $\qquad \varphi_{S_1}(t) = \dfrac{\sin t}{t}$



discontinuous density

f_{S_2} $\qquad \varphi_{S_2}(t) = \left(\dfrac{\sin t}{t}\right)^2$

see page 8



continuous density

f_{S_3} = continuously differentiable density $\qquad \varphi_{S_3}(t) = \left(\dfrac{\sin t}{t}\right)^3$

.../...

(what is the limiting distribution of S_n, as $n \to \infty$?)