THE POISSON PROCESS

I - THE EXPONENTIAL DISTRIBUTION

The exponential distribution models the waiting time until the occurence of an event.

Let $T \in [0, \infty)$ = waiting time to the first event.

$P(\text{'success'}) = \frac{\lambda}{n}, \lambda > 0$ ← By time t we have completed nt trials

$\Rightarrow P(\text{'no success in } [0,t]\text{'}) = \left(1 - \frac{\lambda}{n}\right)^{nt} \to e^{-\lambda t}$

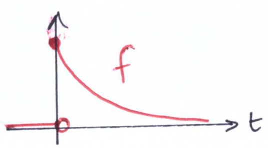
So that

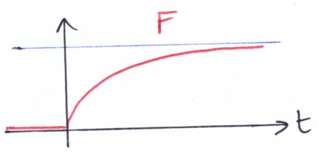
$P(T > t) = e^{-\lambda t}$

$F(t) = P(T \leq t) = 1 - e^{-\lambda t}, \quad t \geq 0$

We say that T has an exponential distribution and we write $T \sim Exp(\lambda)$.

Density is $f(t) = \lambda e^{-\lambda t}, \quad t \geq 0$





<u>Property P1</u> : lack of memory / Memoryless property:

If $T \sim Exp(\lambda)$, then $\forall t, s > 0$,

$$P(T > t+s \mid T > t) = P(T > s)$$

Indeed,

$$P(T > t+s \mid T > t) = \frac{P(T > t+s, \cancel{T > t})}{P(T > t)} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} = e^{-\lambda s}$$

"no ageing for devices whose lifetimes are exponentially distributed": the probability for an electronic component of lasting an additional s hours, given that it lasted t hours, is the same as the probability of lasting s hours when first put into operation.

↘ An important related notion is that of the <u>HAZARD FUNCTION</u> (of failure-rate function), $\lambda(t)$.

$\lambda(t)$ = if T is the life-time of a certain device, and the device has 'survived' for the first t time units of its operation, the $\lambda(t)$ gives the density of the "immediate failure" probability.

$$\lambda(t)\,dt = P(T \in (t, t+dt) \mid T > t)$$
$$= \frac{P(T \in (t, t+dt))}{P(T > t)}$$
$$= \frac{f(t)\,dt}{1 - F(t)} \Rightarrow \boxed{\lambda(t) = \frac{f(t)}{1 - F(t)}}$$

For the exponential distribution,

$\lambda(t) = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda$ = constant failure rate.

<u>Remark</u>: take $\lambda(t) = C t^{\alpha}$

↘ $\alpha = 0$ = constant rate

↘ $\alpha > 0$ = rate increases with time.

Put $G(t) = 1 - F(t) =$ tail of the distribution (3)

$f(t) = F'(t) = -G'(t)$

Then

$$\lambda(t) = \frac{f(t)}{G(t)} = -\frac{G'(t)}{G(t)} = -\frac{d}{dt}\log G(t)$$

and

$$G(t) = \exp\left(-\int_0^t \lambda(u)\,du\right), \quad \lambda(0)=0$$

$$f(t) = \lambda(t)\exp\left(-\int_0^t \lambda(u)\,du\right)$$

Put $\lambda(t) = C x^{\alpha} = \underbrace{\frac{k}{\lambda^k}}_{=C} x^{k-1}$ α

ensures we end up with a density

We get

$$\boxed{f(t) = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} e^{-\left(\frac{x}{\lambda}\right)^k}} = \text{WEIBULL DISTRIBUTION}.$$

Property P2: The minimum of several independently exponentially distributed RVs also has an exponential distribution.

If $T_1, \ldots, T_n$ are i.d, $T_i \sim \text{Exp}(\lambda_i)$

Then $M = \min(T_1, \ldots, T_n) \sim \text{Exp}\left(\sum_{i=1}^n \lambda_i\right)$.

Indeed,

$$\begin{aligned}
P(M>t) &= P(T_1>t, \ldots, T_n>t) \\
&= P(T_1>t) \times \cdots \times P(T_n>t) \\
&= e^{-\lambda_1 t} \times \cdots \times e^{-\lambda_n t} \\
&= e^{-\left(\sum_{i=1}^n \lambda_i\right)t}.
\end{aligned}$$

Application: T_j = lifetimes of independently functioning vital components of a certain device. (4)

$\Rightarrow$ lifetime of the device is also $\sim$ Exp.

Also, we can derive the probability that it was the j-th component which caused the failure of the device.

Take w.l.o.g. $j=n$ and put $\tau = \min(T_1, \ldots, T_{n-1}) \sim \text{Exp}\left(\sum_{i=1}^{n-1} \lambda_i\right)$.

The probability that T_n is the smallest of all the T_j's is:

$$\begin{aligned}
P(\tau > T_n) &= \int_0^\infty P(\tau > T_n \mid T_n = t)\, P(T_n \in dt) \\
&= \int_0^\infty P(\tau > t)\, \lambda_n e^{-\lambda_n t}\, dt \\
&= \int_0^\infty e^{-t\left(\sum_{i=1}^{n-1}\lambda_i\right)} \lambda_n e^{-\lambda_n t}\, dt \\
&= \lambda_n \int_0^\infty \exp\left\{-t\left(\sum_{i=1}^{n}\lambda_i\right)\right\} dt \\
&= \frac{\lambda_n}{\sum_{i=1}^n \lambda_i} \\
&= \text{proportional to the rate } \lambda_n.
\end{aligned}$$

→ The exponential distribution will play a key role in the characterization of the Poisson process. First, we review the (formal) definition of a Stochastic Process (SP)

II_ THE POISSON PROCESS.

II.1. Stochastic processes.

A Stochastic Process (SP) $\{X_t\}_{t\in T}$ is a collection of RVs on a common probability space $(\Omega, \mathcal{F}, P)$. The set T can be

- discrete : $T = \{0, 1, 2, \dots\}$
  $\Rightarrow$ yields TIME SERIES.
- continuous : $T = [0, \infty)$, or $[0, a]$ etc.

The parameter set T is usually called the time; although its meaning may be position in space e.g. when $T \subset \mathbb{R}^d$.

The random process $X = X_t(\omega)$ is a function of two variables t and ω.

(i) When t is fixed, we get a single RV $X_t(.)$, completely characterized by its distribution F_t.

Rk: Knowing the F_ts isn't enough to fully describe the process, since it doesn't tell us anything about the dependence of the X_ts for different times t.

(ii) When ω is fixed, we get a realization (trajectory, path) of the process = function of t.

Pathwise properties of an SP are the properties of these functions.

↳ usually one identifies a functional space of the trajectories of the SP with Ω containing w.p.1 the trajectories of the SP with Ω

Ex: • $\mathcal{C}[0,T]$ = space of continuous functions on $[0,T]$

OK for Brownian motion.

The proba distribution induced on the functional space is called the DISTRIBUTION OF THE PROCESS.

The distribution can be uniquely characterized by the family of FINITE DIMENSIONAL DISTRIBUTIONS (or FDD) = collection of all joint distributions $F_{t_1,..,t_n}(x_1,..,x_n)$ of the RVs $X_{t_1},..,X_{t_n}$, $n \geq 1$, $t_1,..,t_n \in T$.

For any SP, the family of FDD is CONSISTENT in the sense that

$$F_{t_1,t_2,..,t_n,t_{n+1}}(x_1,x_2,..,x_n,\infty) = F_{t_1,..,t_n}(x_1,..,x_n).$$

Standard to define an SP by specifying its FDDs.
And this is a legitimate thing to be doing, since Caratheodori Theorem ensures that for any consistent family of FDDs, there always exist an SP (ie ∃ a distribution on the space of real valued functions $T \to \mathbb{R}$ on T) whose FDD coincide with that family.

Def: An SP $\{X_t\}_{t\in T}$ is STATIONARY if $\forall n \geq 1$ and $t_1,..,t_n \in T$, its FDD $F_{t_1+s,..,t_n+s}$ do not depend on s.

The statistical properties of the process remain unchanged as time elapses → very demanding assumption & hard to check in practice. In general, one deals with:

Def: An SP $\{X_t\}_{t\in T}$ is WEAKLY STATIONARY if

(i) $m_t = E X_t$ is constant (indpt of t)

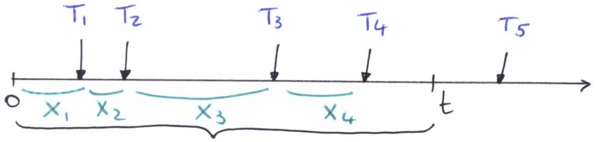
(ii) $r(s,t) = Cov(X_s, X_t) = E(X_s - m_s)(X_t - m_t)$
$= \gamma(t-s)$ depends on time ≠ only.

II. 2. Equivalent definitions of the Poisson Process

The Poisson Process (PoP) is one of the most important models used in queuing theory.

Ex: • arrival process of customers.
• but also, time of defect of an electronic component.

Poisson process = represents discrete arrivals:



$N_t = \#$ arrivals in $(0,t)$. $\quad (N_0 = 0)$

$\Rightarrow$ • $X_i \overset{d}{=} X$ are inter-arrival times

• $T_i = X_1 + \cdots + X_i$ = i-th arrival time.

Information about arrival times can be represented in terms of a COUNTING PROCESS $\{N_t\}_{t\geq 0}$, which counts the number of events that occured in $(0,t)$:

$$\forall n \geq 0, \qquad \{N_t \geq n\} = \{T_n \leq t\}.$$

We give two (equivalent) definitions of the Poisson Process:

Def 1: The counting process $\{N_t\}_{t\geq 0}$ is a POISSON PROCESS with rate $\lambda > 0$ (denoted $PoP(\lambda)$) if

(i) Increments are mutually independent:
$\forall\ 0 \leq s_1 < t_1 \leq s_2 < t_2 \leq \cdots \leq s_k < t_k$
(non-overlapping intervals),
$N_{t_1} - N_{s_1}, \ldots, N_{t_k} - N_{s_k}$ are mutually independent

(ii) $\forall s,t \geq 0, \quad P(N_{t+s} - N_s = n) = \dfrac{(\lambda t)^n}{n!} e^{-\lambda t}$
$n \geq 0$

Def. 2: Let $X_1, \ldots, X_n$ be iid RVs $\sim Exp(\lambda)$, $\lambda > 0$.
Put
$$\begin{cases} T_n = X_1 + \cdots + X_n, & n \geq 1 \\ T_0 = 0 \end{cases}$$
The counting process $\{N_t\}_{t\geq 0}$, defined as $N_t = \max\{n : T_n \leq t\}$ is called a Poisson Process with rate $\lambda > 0$.

Remark: Note that, $\forall n \geq 0\ \forall t > 0 \quad N_t \geq n \iff T_n \leq t$.

• T_n = Sum of n RV, indpt, $Exp(\lambda)$
$\Rightarrow T_n \sim \gamma(n, \lambda)$ [GAMMA distribution]

We show that $N_t = \max\{n : T_n \leq t\} \sim \mathcal{P}(\lambda t)$.

Let $n \geq 0$.

We have
$$P(N_t \geq n) = P(T_n \leq t) = \frac{\lambda^n}{(n-1)!} \int_0^t x^{n-1} e^{-\lambda x}\, dx = \frac{\lambda^n}{(n-1)!} I_n.$$

Integrating by parts yields:
$$I_n = \left[\frac{x^n}{n} e^{-\lambda x}\right]_0^t + \int_0^t \lambda \frac{x^n}{n} e^{-\lambda x}\, dx$$

$e^{-\lambda x}$ $\quad -\lambda e^{-\lambda x}$
x^{n-1} $\quad \dfrac{x^n}{n}$

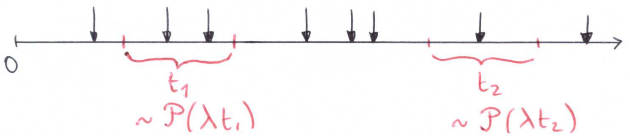
$$\Rightarrow P(N_t \geq n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t} + \underbrace{\frac{\lambda^{n+1}}{n!} I_{n+1}}_{= P(N_t \geq n+1)}$$

$$\Rightarrow P(N_t = n) = P(N_t \geq n) - P(N_t \geq n+1) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

i.e. $N_t \sim \mathcal{P}(\lambda t)$. ∎

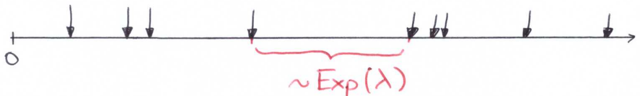
We defined the PoP(λ) in two ways so far:

#1



Number of of arrivals N_t in a finite interval of length t is $\mathcal{P}o(\lambda t)$ & the number of arrivals in non-overlapping intervals are independent.

#2



The inter-arrival times are independent and are $\exp(\lambda)$ distributed

Remarks: (i) To show that #2 $\Rightarrow$ #1, it remains to show that increments of $\{N_t\}_{t\geq 0}$ are independent and stationary (property (ii) in definition 1 was established on page 8 with $s=0$ and t arbitrary).

(ii) We show that #1 $\Rightarrow$ #2. Essentially, we need to show that variables $X_1, X_2, \dots, X_n$ are iid, $\mathrm{Exp}(\lambda)$.

↘ First, note that $\{X_1 > t\}$ iff. no arrival in $[0,t]$, ie.

$$P(X_1 > t) = P(N_t = 0) \underset{\uparrow}{=} e^{-\lambda t} \quad \Rightarrow X_1 \sim \mathrm{Exp}(\lambda)$$

(prop (ii) in #1)

↘
$$\begin{aligned} P(X_2 > t) &= E\, 1(X_2 > t) \\ &= E\, E\{1(X_2 > t) \mid X_1\} \\ &= E\, P(X_2 > t \mid X_1) \\ &= \int_0^\infty P(X_2 > t \mid X_1 = u)\, \lambda e^{-\lambda u}\, du. \end{aligned}$$

Now,
$$\begin{aligned} P(X_2 > t \mid X_1 = s) &= P(\text{no arrivals in } (s, s+t) \mid X_1 = s) \\ &= P(\text{no arrivals in } (s, s+t)) \quad \leftarrow \text{indpce of increments} \\ &= P(N_{t+s} - N_s = 0) \quad \leftarrow \text{stationary increments} \\ &= e^{-\lambda t} \end{aligned}$$

$\Rightarrow$ $X_2 \sim \mathrm{Exp}(\lambda)$, and is independent of X_1.
The argument must now be repeated for all $n > 2$.

(iii) In fact, there is a third equivalent definition of PoP(λ):

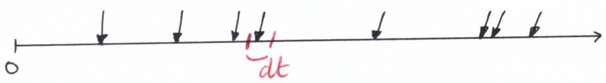
Def 3: The counting process $\{N_t\}_{t\geq 0}$ is a Poisson Process with rate $\lambda > 0$ if

(i) Increments are independent & stationary

(ii) $P(N_h = 1) = \lambda h (1 + o(1))$.
$P(N_h \geq 2) = o(h)$.

Markov view of PoP: PoP = continuous time Markov Process.

#3



In an infinitesimal time interval dt, there may occur only one arrival; which happens with proba λdt; independently of arrivals outside this interval.

Heuristics: the Poisson distribution arises quite naturally from definition #3: subdivide interval $(0,t)$ into n subintervals $[k\frac{t}{n}, (k+1)\frac{t}{n})$; each of length $\frac{t}{n}$.

Proba of one arrival in one of these intervals is $\lambda \frac{t}{n}$ (def 3)

Since arrivals in each of these intervals are independent (property (i) in # 3), we see that $N_t \sim Bi\left(n, \frac{\lambda t}{n}\right)$.

$\approx P(\lambda t)$

This argument can be made precise. Good.

II. 3. Properties of the Poisson Process.

- The renewal function is defined as the expected value of the number of arrivals in $[0,t]$:

$\forall t \geq 0, \quad m(t) = E(N_t)$.

- Clearly, for the PoP(λ), $\boxed{m(t) = \lambda t}$

Remark: For arbitrary waiting time distribution; $X_i \sim F$, denoting by F_n the distribution of T_n;

$F_n(x) = \int_0^\infty F_{n-1}(x-u)\, dF(u)$; $F_1(x) = F(x)$,
(convolution formula)

we have that

$$m(t) = \mathbf{E}\, N_t = \sum_{n=0}^{\infty} P(N_t > n) = \sum_{n=1}^{\infty} P(N_t \geq n)$$
$$= \sum_{n=1}^{\infty} F_n(t)$$

Since $\{N_t \geq n\} = \{T_n \leq t\}$.

= sum of cdfs.

$\Rightarrow$ The renewal function is non-decreasing, right continuous, and $\lim_{t\to\infty} m(t) = +\infty$.

Moreover, $\forall t \geq 0$, $m(t) = \sum_{n\geq 1}^{\infty} F_n(t) < \infty$

(requires a proof, but we won't do it).

Theorem. Let $\{N_t\}$ and $\{M_t\}$ be two independent PoP with rates λ and μ respectively. Then the sum $L_t = M_t + N_t$ is a PoP with rate $(\lambda+\mu)$.

Remains true for the sum of any number of independent PoP.

Proof: By independence, for any $t > 0$, $N_t + M_t \sim P((\lambda+\mu)t)$. Also, $\{L_t\}$ has independent increments since both $\{N_t\}$ and $\{M_t\}$ have this property and are independent of each other. ∎

Ex: A shop has two entrances, one from XY street and the other from YX street. Flows of customers arriving in the shop from these two entrances are independent Poisson processes with rates $\lambda_1 = 0.5\ \text{min}^{-1}$ and $\lambda_2 = 1.5\ \text{min}^{-1}$, respectively.

(i) What is the probability that no new customers will enter the shop during a fixed three-minute interval?

(ii) What is the mean time between arrivals of new customers?

(iii) What is the probability that the first customer entered from XY street?

(i) Arrival of customers to the shop is described by the sum of two independent PoP & hence is a PoP itself, with rate $\lambda = \lambda_1 + \lambda_2 = 2\ (\text{min}^{-1})$.

$\Rightarrow$ Inter-arrival times are $X_j \sim Exp(2)$, so that $P(X_1 > 3) = e^{-2\times 3} \approx 0.0025$.

(ii) Mean inter-arrival time is $1/\lambda = 0.5$ min.

(iii) From page 4, the probability is given by $\frac{\lambda_1}{\lambda_1+\lambda_2} = 0.25$.

Rk: In fact, this probability is also the proba that a given customer enters from XY street (and not just the first one).

We now consider the 'inverse' situation:

Theorem: Let $\{N_t\}$ be a PoP(λ). Suppose that in $\{N_t\}$, each jump is marked independently w.p $p \in [0,1]$, and denote by M_t the number of 'marked' arrivals by time t.
Then processes $\{M_t\}$ and $\{N_t - M_t\}$ are indpt PoP with rates λp and $(1-p)\lambda$, respectively.

proof: Writing $N_t^1 = M_t$ and $N_t^2 = N_t - M_t$ for convenience, we have

- $$P(N_t^1 = k, N_t^2 = \ell) = E\{P(N_t^1 = k, N_t^2 = \ell \mid N_t)\}$$

$$= \underbrace{P(N_t^1 = k, N_t^2 = \ell \mid N_t = k+\ell)} \times \underbrace{P(N_t = k+\ell)}_{\frac{(\lambda t)^{k+\ell}}{(k+\ell)!} e^{-\lambda t}}$$

There are $\binom{k+\ell}{k}$ ways to mark k jumps out of $(k+\ell)$
$\Rightarrow P(\dots) = \binom{k+\ell}{k} p^k (1-p)^\ell$

$$= \frac{\cancel{(k+\ell)!}}{k!\,\ell!} p^k (1-p)^\ell \frac{(\lambda t)^{k+\ell}}{\cancel{(k+\ell)!}} e^{-\lambda(p+1-p)t}$$

$$= \underbrace{\frac{(\lambda p t)^k}{k!} e^{-\lambda p t}} \underbrace{\frac{[\lambda(1-p)t]^\ell}{\ell!} e^{-\lambda(1-p)t}}$$

Joint Distribution factorizes $\Rightarrow$ N_t^1 and N_t^2 are independent RVs and $\begin{cases} N_t^1 \sim P(p\lambda t) \\ N_t^2 \sim P((1-p)\lambda t) \end{cases}$.

- likewise, can show that increments $(N_{t+s}^1 - N_s^1)$ and $(N_{t+s}^2 - N_s^2)$ are $P(p\lambda t)$ and $P((1-p)\lambda t)$, respectively.
- It remains to show that increments of $\{N_t^1\}$ and $\{N_t^2\}$ are independent.

Let $s, t > 0$. We have

$$P(N_s^1 = k, N_{t+s}^1 - N_s^1 = \ell) =$$
$$E\{P(N_s^1 = k, N_{t+s}^1 - N_s^1 = \ell \mid N_s, N_{t+s} - N_s)\}$$

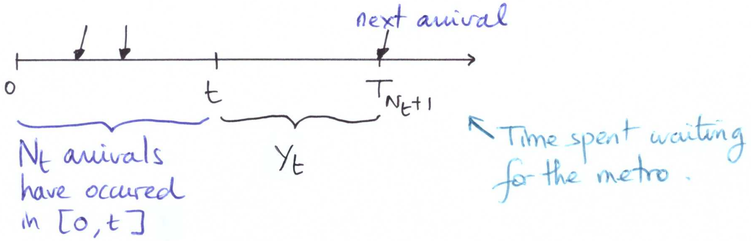
By conditioning on N_s and $N_{t+s} - N_s$, we get rid of the randomness associated with the number of jumps occuring between time s and $(t+s)$ $\Rightarrow$ only randomness on the marking remains. And marking is done independently each time! $\Rightarrow$

$$= E\{P(N_s^1 = k \mid N_s, \cancel{N_{t+s} - N_s}) P(N_{t+s}^1 - N_s^1 = \ell \mid \cancel{N_s}, N_{t+s} - N_s)\}$$
$$= E\{P(N_s^1 = k \mid N_s) P(N_{t+s}^1 - N_s^1 = \ell \mid N_{t+s} - N_s)\}$$
$$= E\{P(N_s^1 = k \mid N_s)\} E\{P(N_{t+s}^1 - N_s^1 = \ell \mid N_{t+s} - N_s)\}$$
$$= P(N_s^1 = k) P(N_{t+s}^1 - N_s^1 = \ell).$$ ∎

- A few more definitions:

(i) Residual time $Y_t = T_{N_t+1} - t$

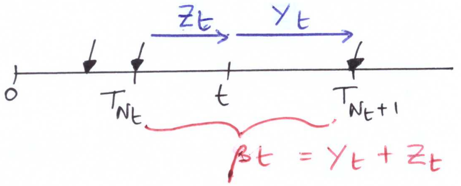
$= $ time until the next arrival.



(ii) Age process $Z_t = t - T_{N_t}$

$=$ elapsed time since last arrival

(iii) Duration of an inter-arrival interval $\beta_t = T_{N_t+1} - T_{N_t}$



↳ For a $PoP(\lambda)$, the residual time is such that

$$\forall x \geq 0, \quad P(Y_t > x) = P(N_{t+x} - N_t = 0) = e^{-\lambda x}.$$

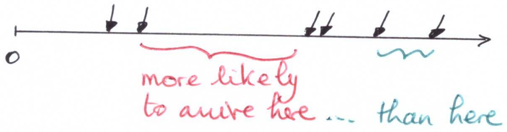
i.e. $Y_t \sim Exp(\lambda) \Rightarrow$ Mean waiting time is still $1/\lambda$.

Seems paradoxical at first sight since the residual time has the same distribution as inter-arrival times; while residual time is always smaller than the duration between two arrivals.

↳ This is the magic of the exponential distribution; & its lack of memory property.

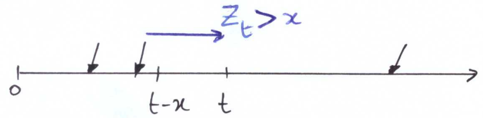
↳ In words, suppose that cars are passing a point of the road according to a PoP; and that the mean interval between two cars is 10 mins. A hitchhicker arrives at a random instant. The mean waiting time until the next car is still 10 mins.

Why? Because the probability that a hitchhicker arrives during a long inter-arrival interval is greater than during a short interval.



↳ The age process cannot be larger than t since $Z_t = t - T_{N_t}$.

$$\forall x < t, \quad P(Z_t > x) = P(N_t - N_{t-x} = 0) = e^{-\lambda x}$$

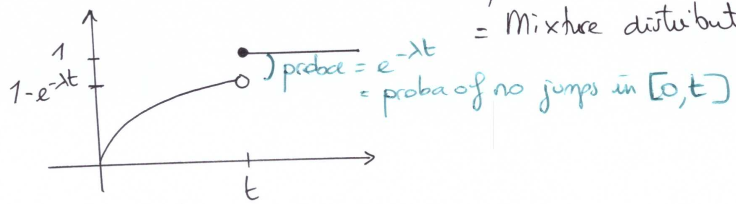


Between times $(t-x)$ and t, no arrivals must have occured; i.e. $N_{t-x} = N_t$.

Thus $F_{Z_t}(x) = \begin{cases} 1 - e^{-\lambda x} & \text{if } 0 \leq x < t \\ 1 & \text{if } x \geq t \end{cases}$

$=$ truncated exponential distribution.

$=$ Mixture distribution



$$EZ_t = \int_0^t \lambda x e^{-\lambda x} dx + t\, P(Z_t = t)$$

$$= \left[-x e^{-\lambda x}\right]_0^t + \int_0^t e^{-\lambda x} dx + t e^{-\lambda t}$$

$$= \frac{1 - e^{-\lambda t}}{\lambda}.$$

⇒ The expected value of the current inter-arrival interval is thus

$$E\beta_t = EY_t + EZ_t = \frac{1}{\lambda} + \frac{1-e^{-\lambda t}}{\lambda}$$

$$= \frac{2-e^{-\lambda t}}{\lambda} \longrightarrow \frac{2}{\lambda} \quad \text{as } t\to\infty$$

Always larger than $\frac{1}{\lambda} = EX_1$!

For large t, the current inter-arrival time (seen by an observer at time t) is twice as long as the average inter-arrival time.

II.4 A few more observations & extensions.

Recall the definition of the PoP on page 8:

$$N_t = \max\{n \mid T_n \leq t\} \; ;$$

where • $T_n = X_1 + \dots + X_n, \quad n \geq 1$

• $X_i \sim Exp(\lambda)$ [what is said in this subsection holds true for general distribution of X_i]

We are interested in what happens to the PoP in the long run.

→ $P(N_t = n) = P(N_t \geq n) - P(N_t \geq n+1)$

$= P(T_n \leq t) - P(T_{n+1} \leq t)$

$= F_n(t) - F_{n+1}(t)$,

where $F_n(t) = F^{(n)}(t)$ = n-fold convolution of F with itself.

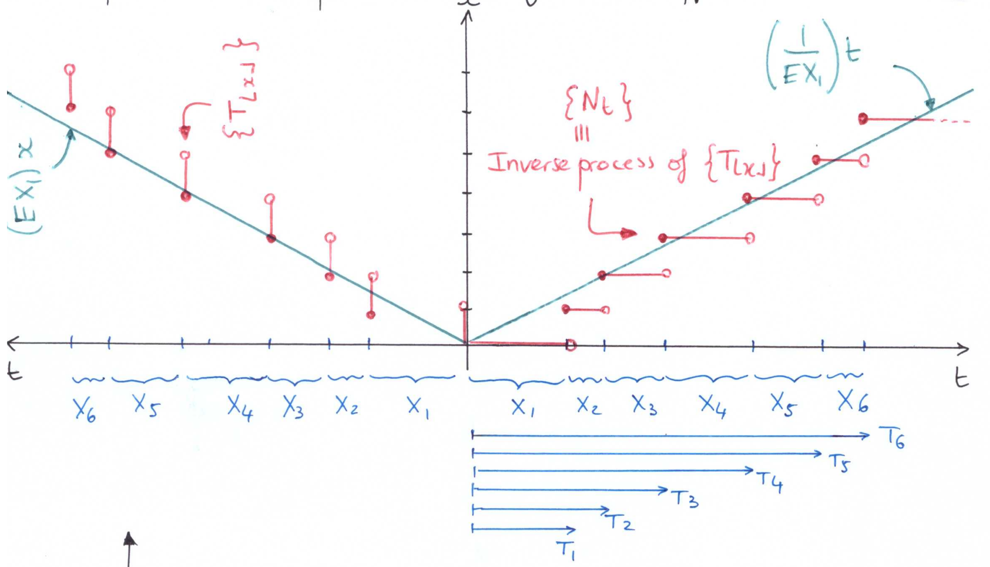
very tedious to compute.

→ The long term behaviour of N_t can be understood as a simple consequence of the SLLN.

Consider the continuous time process $T_{\lfloor x\rfloor} = X_1 + \dots + X_{\lfloor x\rfloor}$,

$x > 0$.

A picture will help understanding what happens:



Note that $T_{\lfloor x\rfloor} = X_1 + \dots + X_{\lfloor x\rfloor}$

$\Rightarrow \frac{T_{\lfloor x\rfloor}}{x} \xrightarrow{a.s.} EX_1$ (SLLN)

ie: $T_{\lfloor x\rfloor} \simeq (EX_1)\,x$

= lies along a straight line with slope EX_1. (in green)

⇒ $\{N_t\}$, being the generalized inverse of the process $\{T_{\lfloor x\rfloor}\}$, will also be close to a straight line for t large; but with the reciprocal slope $\frac{1}{EX_1}$.

(look: axes t and x are reversed!)

This can actually be made rigorous, and we have:

Holds for non-neg. RVs X_1

$$\boxed{\frac{N_t}{t} \xrightarrow{a.s.} \frac{1}{EX_1} \quad \text{as } t\to\infty}$$

The case $EX_1 = \infty$ is not excluded

Rk: For a PoP(λ), $\frac{N_t}{t} \xrightarrow{a.s.} \lambda$ as $t\to\infty$

Rk: • As it is the case for RVs, the SLLN can be refined by the CLT when the second moments are finite (which is, of course, true for the PoP(λ)). If $\sigma^2 = \mathrm{Var}\, X_j < \infty$, then ($\mu = E X_j$)

$$\frac{N_t - t/\mu}{\sigma\sqrt{t/\mu^3}} \xrightarrow{d} \mathcal{N}(0,1), \quad \text{as } t \to \infty$$

(the convergence is actually uniform in the sense that

$$\sup_x \left| P\left(\frac{N_t - t/\mu}{\sigma\sqrt{t/\mu^3}} \leq x \right) - \Phi(x) \right| \to 0 \quad \text{as } t \to \infty.)$$

• As stated on the previous page, $\frac{N_t}{t} \xrightarrow{a\text{-}s} \frac{1}{\mu}$. It turns out that expectations of the left-hand side converge to the same limit. This result is known as the RENEWAL THEOREM:

$$\boxed{\lim_{t\to\infty} \frac{m(t)}{t} = \frac{1}{\mu}}$$

(recall that $m(t) := E N_t$)

This result is obvious for a PoP(λ) since $m(t) = \lambda t = t/\mu$; $\mu = EX_1 = \frac{1}{\lambda}$.