Problem 1.
Let $X_1, \ldots, X_n$ be a random sample from the $\mathcal{N}(\mu, 1)$ distribution. We compare two models,

$$\mathcal{F}_0 : \mathcal{N}(0,1)\,, \quad \text{and} \quad \mathcal{F}_1 : \mathcal{N}(\mu,1)\,.$$

(i) Test the hypothesis that $\mu = 0$ versus a two-sided alternative. Show that at the 95% confidence level, you reject the null if

$$|\bar{X}| > \frac{1.96}{\sqrt{n}}\,.$$

(ii) Derive a similar rejection region based on AIC.

(iii) Derive a similar rejection region based on BIC.

Problem 2.
Consider two datasets, each having k categories, and that the number of observations in each category is given as follows

Category	1	2	$\ldots$	k
dataset 1	n_1	n_2	$\ldots$	n_k
dataset 2	m_1	m_2	$\ldots$	m_k

where the total number of observations are $n_1 + \ldots + n_k = n$ and $m_1 + \ldots + m_k = m$, respectively. We assume that the observations have a multinomial distribution, given by

$$\mathbf{P}(n_1, \ldots, n_k) = \frac{n!}{n_1! \ldots n_k!}\, p_1^{n_1} \ldots p_k^{n_k}\,,$$
$$\mathbf{P}(m_1, \ldots, m_k) = \frac{n!}{m_1! \ldots m_k!}\, q_1^{m_1} \ldots q_k^{m_k}\,,$$

where $p_j > 0$ and $q_j > 0$ denote the probabilities that each event in the first and second dataset results in the category j.

(i) Write down the log-likelihood of the model consisting of two individual models for the two datasets (Model I). How many free parameters are there? Derive the AIC for this model.

(ii) Write down the log-likelihood, if we now assume that the two distributions are equal (Model II). How many free parameters? Derive the AIC for this model.

The following table shows two datasets each having 5 categories.

Category	1	2	3	4	5
Data set 1	304	800	400	57	323
Data set 2	174	509	362	80	214

(iii) Obtain the maximum likelihood estimates of the parameters for Model I and II. Based on the AIC, which model would you pick?

Problem 3. *Laplace Approximation*
Consider the function

$$f(x) = x^{\alpha-1}e^{-\beta x}\,,$$

where $x > 0$, $\alpha > 1$ and $\beta > 0$. Find the Laplace approximation to

$$\mathcal{I} = \int_0^\infty f(x)dx\,.$$